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Proof of the Fukui conjecture via resolution of singularities and related methods. II☆

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☆*Dedicated to the memory of Prof. Kenichi Fukui (1918–1998).*

The present article is a direct continuation of the first part of this series. We reduce a proof of the Fukui conjecture (concerning the additivity problem of the zero-point vibrational energies of hydrocarbons) to that of a proposition related to the theory of algebraic curves, so that we can focus on the key mechanism of the additivity phenomena. Namely, by establishing what is called the Basic Piecewise Monotone Theorem (BPMT), we reduce a proof of the Fukui conjecture to that of a proposition, called the Local Analyticity Proposition, Version 1 (LAP1), which admits a proof via resolution of singularities. By LAP1, the essential part of the mechanism of the "asymptotic linearity phenomena" is extracted and is elucidated by using tools from the mathematical theory of algebraic curves, whose language is of vital importance in analyzing the crux of the additivity mechanism.

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1. Introduction

This article is the second part of a series devoted to extending the foundation of the Asymptotic Linearity Theorems (ALTs), which prove the Fukui conjecture concerning the additivity problem of the zero-point vibrational energies of hydrocarbons. The conjecture continues to be of vital significance in the recent development of the theory of generalized repeat space $\mathscr{X}_r(q, d)$. (The reader is referred to [1] and references therein.)

In part I [1] of this series of articles, in conjunction with the repeat space theory (RST) (cf. [2–8] and references therein), we established the following sequence of logical implications.

PML (Piecewise Monotone Lemma) $\Rightarrow$ $\mathscr{G}$ Boundedness Theorem $\Rightarrow$ Special Functional ALT $\Rightarrow$ Functional ALT $\Rightarrow$ the Fukui conjecture.

In section 2 of the present part II, we formulate a problem in which we ask if we can reduce a proof of the $\mathscr{G}$ Boundedness Theorem to that of proposition 1.1 (Local Analyticity Proposition, Version 1, LAP1) given at the end of section 2. If the problem can be solved affirmatively, then the essential part of the mathematical mechanism that causes the "asymptotic linearity phenomena" is extracted and can be elucidated by using tools from the mathematical theory of algebraic curves, whose language is of vital importance in analyzing the crux of the additivity mechanism.

By applying the theory of algebraic curves and by recalling techniques used in perturbation theory [9], it is seen that this proposition 1.1 is true, admitting a proof via resolution of singularities. By establishing what is called the Basic Piecewise Monotone Theorem (BPMT), we solve the above question affirmatively (cf. section 4), so we established the following sequence of logical implications:

LAP1 $\Rightarrow$ PML2 $\Rightarrow$ $\mathscr{G}$ Boundedness Theorem $\Rightarrow$ Special Functional ALT $\Rightarrow$ Functional ALT $\Rightarrow$ the Fukui conjecture.

Here, PML2 stands for Piecewise Monotone Lemma Version 2, which is an enhanced version of the PML and is indispensable for a broader range of applications of the RST. A detailed proof of the LAP1 via resolution of singularities in conjunction with the RST shall be published in part III of this series.

2. Formulation of the problem of reduction

Before formulating the problem of reduction mentioned in section 1, we need to recall the definition (Definition 2.1) of symbols from [1] and introduce some more (Definition 2.2) for this section and the subsequent sections in the present article.

Throughout, let $\mathbb{Z}^+$, $\mathbb{R}$ denote, respectively, the set of all positive integers and the set of all real numbers.

Definition 2.1. Let S_1 and S_2 be nonempty subsets of $\mathbb{R}$. A function $f\colon S_1 \to S_2$ is said to be *nondecreasing* if $x_1 \leqslant x_2$ implies $f(x_1) \leqslant f(x_2)$ for all $x_1, x_2 \in S_1$. A function $f\colon S_1 \to S_2$ is said to be *nonincreasing* if $x_1 \leqslant x_2$ implies $f(x_2) \leqslant f(x_1)$ for all $x_1, x_2 \in S_1$. A function $f\colon S_1 \to S_2$ is said to be *monotone* if it is either nondecreasing or nonincreasing.

Let $a, b \in \mathbb{R}$ with $a < b$ and let $I = [a, b]$. A function $f\colon I \to \mathbb{R}$ is said to be *piecewise monotone* if there exists a finite partition

$$a = x_0 < x_1 < \cdots < x_n = b \quad (n \in \mathbb{Z}^+) \tag{2.1}$$

of the interval I such that the restriction $f|[x_{i-1}, x_i]$ is monotone for all $i \in \{1, \ldots, n\}$. In this case, f is said to have an n-partition of monotonicity.

A real-valued function on a subset $S \subset \mathbb{R}$ is called *real analytic on S* if it is the restriction to S of a function which is real analytic on some open set $O \supset S$.

Let $a, b \in \mathbb{R}$ with $a < b$ and let $I = [a, b]$.
If $f\colon I \to \mathbb{R}$ is piecewise monotone, let

$$\mathrm{Mo}(f) := \min\,\{n \in \mathbb{Z}^+\colon f \text{ has an } n\text{-partition of monotonicity}\}. \tag{2.2}$$

The $\mathrm{Mo}(f)$ is called the *monotonicity number of f*.
If $f\colon I \to \mathbb{R}$ is not piecewise monotone, let

$$\mathrm{Mo}(f) = \infty. \tag{2.3}$$

$C^\omega(I)$: the ring (UFD) of all real analytic functions defined on I.
$C^\omega(I)[\lambda]$: the polynomial ring (UFD) over $C^\omega(I)$ in the indeterminate λ.
$C(I)$: the ring of all real-valued continuous functions defined on I.
$C(I)[\lambda]$: the polynomial ring over $C(I)$ in the indeterminate λ.
$\mathbb{R}[\lambda]$: the polynomial ring (UFD) over $\mathbb{R}$ in the indeterminate λ.
For each $\theta \in I$, let $\mathrm{Ev}_\theta\colon C(I)[\lambda] \to \mathbb{R}[\lambda]$ be the ring homomorphism defined by

$$\mathrm{Ev}_\theta(c_0\lambda^n + c_1\lambda^{n-1} + \cdots + c_n) = c_0(\theta)\lambda^n + c_1(\theta)\lambda^{n-1} + \cdots + c_n(\theta). \tag{2.4}$$

$V_I(\varphi)$: the total variation of a real-valued function φ on I, i.e.,

$$V_I(\varphi) = \sup_\Delta \sum_{i=1}^{n} |\varphi(t_i) - \varphi(t_{i-1})|. \tag{2.5}$$
$$(\Delta\colon a = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = b)$$

$CBV(I)$: the normed space of all real-valued continuous functions of bounded variation on I equipped with the norm given by

$$\|\varphi\| = \sup\{|\varphi(t)|\colon t \in I\} + V_I(\varphi). \tag{2.6}$$

Definition 2.2. Let $a, b \in \mathbb{R}$ with $a < b$ and let $I = [a, b]$.
Map $(I \times \mathbb{R}, \mathbb{R})$: the ring of all functions $f\colon I \times \mathbb{R} \to \mathbb{R}$.
$C^{\omega *}(I)$: the ring defined by $C^{\omega *}(I) := \{f \in C(I)\colon f$ is real analytic in the interior $]a, b[$ of $I\}$.
$C^{\omega *}(I)[\lambda]$: the polynomial ring over $C^{\omega *}(I)$ in the indeterminate λ.
$C^{PM}(I) := \{f \in C(I)\colon f$ is piecewise monotone on $I\}$.

Let Fn: $C(I)[\lambda] \to \mathrm{Map}(I \times \mathbb{R}, \mathbb{R})$ be the ring homomorphism defined by

$$\mathrm{Fn}(c_0\lambda^n + c_1\lambda^{n-1} + \cdots + c_n) = f, \tag{2.7}$$

where $f\colon I \times \mathbb{R} \to \mathbb{R}$ is the function defined by

$$f(\theta, \lambda) = c_0(\theta)\lambda^n + c_1(\theta)\lambda^{n-1} + \cdots + c_n(\theta). \tag{2.8}$$

If X and Y are nonempty sets and $f\colon X \to Y$ is a mapping, $\Gamma(f)$ denotes the graph of f:

$$\Gamma(f) = \{(x, f(x)) \in X \times Y\colon x \in X\}. \tag{2.9}$$

If n is a positive integer, S_n denotes the group of all bijections $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$, i.e., the group of permutations of the set of n elements.
If $a, b \in \mathbb{R}$ with $a < b$, let $H_r(a, b)$ denote the set of all real-valued real analytic functions defined on the interval $]a, b[$.

At this moment, the reader is referred to [1] for the proof of the following $\mathcal{G}$ Boundedness Theorem. The proof used proposition 3.1 given at the beginning of section 3 and the PML1 reproduced below. After referring to the proof in [1], it is easy to see that theorem 2.1 follows directly from the PML2, which is given after the PML1.

Theorem 2.1 ($\mathcal{G}$ Boundedness Theorem, $\mathcal{G}$BT). Let $\tilde{a}, \tilde{b} \in \mathbb{R}$ with $\tilde{a} < \tilde{b}$ and let $\tilde{I} = [\tilde{a}, \tilde{b}]$. Let $p \in C^{\omega}(\tilde{I})[\lambda]$ be a monic polynomial of degree $q \in \mathbb{Z}^+$ given by

$$p = \lambda^q + c_1\lambda^{q-1} + \cdots + c_q. \tag{2.10}$$

Suppose that for any $\theta \in \tilde{I}$, the polynomial

$$\mathrm{Ev}_\theta(p) = \lambda^q + c_1(\theta)\lambda^{q-1} + \cdots + c_q(\theta) \tag{2.11}$$

over the field $\mathbb{R}$ has q real roots. Define the mapping $f\colon \tilde{I} \to \mathbb{R}[\lambda]$ by

$$f(\theta) = \mathrm{Ev}_\theta(p), \tag{2.12}$$

and let $r_j(f(\theta))$ denote the jth root of $f(\theta)$ counted with multiplicity, arranged in the increasing order, where $j \in \{1, \dots, q\}$. Let $a, b \in \mathbb{R}$ with $a < b$ and let $I = [a, b]$. Suppose that I contains all the roots of $f(\theta)$ for all $\theta \in \tilde{I}$. Then, the following statements are true:

(i) For each $\varphi \in CBV(I)$, the function $\theta \mapsto \sum_{j=1}^{q} \varphi(r_j(f\theta)))$ defined on $\tilde{I}$ is real-valued continuous and of bounded variation, i.e., an element of $CBV(\tilde{I})$.

(ii) Define the linear operator $\mathscr{G}$: $CBV(I) \to CBV(\tilde{I})$ by

$$\mathscr{G}(\varphi)(\theta) = \sum_{j=1}^{q} \varphi(r_j(f(\theta))). \tag{2.13}$$

Then, $\mathscr{G}$ is bounded:

$$\|\mathscr{G}\| < \infty. \tag{2.14}$$

Lemma 2.1 (Piecewise Monotone Lemma, Version 1, PML1). Let $a, b \in \mathbb{R}$ with $a < b$ and let $I = [a, b]$. Let $p \in C^{\omega}(I)[\lambda]$ be a monic polynomial of degree $q \in \mathbb{Z}^{+}$ given by

$$p = \lambda^q + c_1\lambda^{q-1} + \cdots + c_q. \tag{2.15}$$

Suppose that for any $\theta \in I$, the polynomial

$$\mathrm{Ev}_{\theta}(p) = \lambda^q + c_1(\theta)\lambda^{q-1} + \cdots + c_q(\theta) \tag{2.16}$$

over the field $\mathbb{R}$ has q real roots, which we denote by $\lambda_1(\theta) \leqslant \lambda_2(\theta) \leqslant \cdots \leqslant \lambda_q(\theta)$. Then, all the λ_j's are piecewise monotone, i.e.,

$$\mathrm{Mo}(\lambda_j) < \infty \tag{2.17}$$

for all $j \in \{1, \dots, q\}$.

Lemma 2.2 (Piecewise Monotone Lemma, Version 2, PML2). Let $a, b \in \mathbb{R}$ with $a < b$ and let $I = [a, b]$. Let $p \in C^{\omega}(I)[\lambda]$ be a monic polynomial of degree $q \in \mathbb{Z}^{+}$ given by

$$p = \lambda^q + c_1\lambda^{q-1} + \cdots + c_q. \tag{2.18}$$

Suppose that for any $\theta \in I$, the polynomial

$$\mathrm{Ev}_{\theta}(p) = \lambda^q + c_1(\theta)\lambda^{q-1} + \cdots + c_q(\theta) \tag{2.19}$$

over the field $\mathbb{R}$ has q real roots. Consider p as an element of $C^{\omega *}(I)[\lambda]$, then p can be factored into first degree monic polynomials:

$$p = (\lambda - d_1)(\lambda - d_2) \cdots (\lambda - d_q), \tag{2.20}$$

where

$$d_1, \ldots, d_q \in C^{\omega *}(I) \bigcap C^{\mathrm{PM}}(I). \tag{2.21}$$

We are ready to formulate our problem of reduction:

Problem S1: It is easy to see that

$$\mathrm{PML2} \Rightarrow \mathscr{G}\mathrm{BT}, \tag{2.22}$$

hence that

$$\mathrm{PML2} \Rightarrow \mathscr{G}\mathrm{BT} \Rightarrow \mathrm{ALT} \Rightarrow \text{Fukui conjecture}. \tag{2.23}$$

(We also know that PML2 is useful for extending alpha existence theorem in the RST.) If PML2 is true, obviously the following Proposition 2.1 (LAP1) is true. Our problem is:

Is it possible to reduce the proof of the $\mathscr{G}$BT to that of the LAP1?

Proposition 2.1 (Local Analyticity Proposition, Version 1, LAP1). Let $a, b \in \mathbb{R}$ with $a < b$ and let $I = [a, b]$. Let $p \in C^{\omega}(I)[\lambda]$ be a monic polynomial of degree $q \in \mathbb{Z}^+$ given by

$$p = \lambda^q + c_1\lambda^{q-1} + \cdots + c_q. \tag{2.24}$$

Suppose that for any $\theta \in I$, the polynomial

$$\mathrm{Ev}_\theta(p) = \lambda^q + c_1(\theta)\lambda^{q-1} + \cdots + c_q(\theta) \tag{2.25}$$

over the field $\mathbb{R}$ has q real roots.

Define $f\colon I \times \mathbb{R} \to \mathbb{R}$ by

$$f(\theta, \lambda) = \lambda^q + c_1(\theta)\lambda^{q-1} + \cdots + c_q(\theta). \tag{2.26}$$

Then, for any $(\theta, \lambda) \in f^{-1}(0) \cap (]a, b[\times \mathbb{R})$ there exist $\varepsilon, \delta > 0, n \in \mathbb{Z}^+$, and $h_1, \ldots, h_n \in H_r(\theta - \varepsilon, \theta + \varepsilon)$ with $h_1(\theta) = \cdots = h_n(\theta) = \lambda$ such that

$$f^{-1}(0) \cap (]\theta - \varepsilon, \theta + \varepsilon[\times]\lambda - \delta, \lambda + \delta[) = \bigcup_{i=1}^{n} \Gamma(h_i). \tag{2.27}$$

3. The glueing tools

For the affirmative solution of the problem of reduction formulated in section 2, we need the following five tools for "glueing locally defined function fragments" (cf. section 4) to construct functions in $C^{\omega *}(I)$ appearing in PML2.

Proposition 3.1 (Glueing Tool 1). Let $a, b \in \mathbb{R}$ with $a < b$ and let $I = [a, b]$. Let $p \in C(I)[\lambda]$ be a monic polynomial of degree $q \in \mathbb{Z}^+$ given by

$$p = \lambda^q + c_1\lambda^{q-1} + \cdots + c_q. \tag{3.1}$$

Suppose that for any $\theta \in I$, the polynomial

$$\mathrm{Ev}_\theta(p) = \lambda^q + c_1(\theta)\lambda^{q-1} + \cdots + c_q(\theta) \tag{3.2}$$

over the field $\mathbb{R}$ has q real roots, which we denote by $\lambda_1(\theta) \leqslant \lambda_2(\theta) \leqslant \cdots \leqslant \lambda_q(\theta)$. Then, all the λ_j's are continuous, i.e.,

$$\lambda_j \in C(I) \tag{3.3}$$

for all $j \in \{1, \ldots, q\}$.

Proof. This is equivalent to proposition 2.1 in [1]. □

Proposition 3.2 (Glueing Tool 2). Let I be a nonempty connected subset of $\mathbb{R}$, let f and $g_1, \ldots, g_n$: $I \to \mathbb{R}$ be continuous functions. Suppose that

(I) $\Gamma(f) \subset \bigcup_{i=1}^{n} \Gamma(g_i)$,

(II) $\Gamma(g_1), \ldots, \Gamma(g_n)$ are pairwise disjoint.

Then, there exists a unique $i \in \{1, \ldots, n\}$ such that

$$\Gamma(f) = \Gamma(g_i). \tag{3.4}$$

Proof. For each $i \in \{1, \ldots, n\}$, define subset E_i of I by

$$E_i = \{x \in I\colon f(x) = g_i(x)\}. \tag{3.5}$$

We first claim that

(i) $\bigcup_{i=1}^{n} E_i = I$,

(ii) $E_1, \ldots, E_n$ are pairwise disjoint,

(iii) E_i is clopen (closed and open) in I for each $i \in \{1, \ldots, n\}$.

Claim (i) follows from the definition of E_i and hypothesis (I), claim (ii) follows from hypothesis (II). To see that claim (iii) is true, consider for each $i \in \{1, \dots, n\}$ the continuous mapping $u_i\colon I \to \mathbb{R} \times \mathbb{R}$ defined by

$$u_i(x) = (f(x), g_i(x)). \tag{3.6}$$

Now notice that the diagonal set

$$\Delta = \{(y, y) \in \mathbb{R} \times \mathbb{R}\colon y \in \mathbb{R}\} \tag{3.7}$$

is closed in $\mathbb{R} \times \mathbb{R}$ and that E_i is expressed as the inverse image of Δ under u_i:

$$E_i = u_i^{-1}(\Delta). \tag{3.8}$$

Thus, E_i is closed in I for each $i \in \{1, \dots, n\}$. By claims (i) and (ii), we immediately notice that E_i is also open in I for each $i \in \{1, \dots, n\}$.

By the hypothesis of the proposition, I is a nonempty connected subset of $\mathbb{R}$, thus clopen subset E_i of I is either $\emptyset$ or I itself. Hence, by claims (i) and (ii), there exists a unique $i \in \{1, \dots, n\}$ such that

$$E_i = I. \tag{3.9}$$

The conclusion follows. □

Proposition 3.3 (Glueing Tool 3). Let I be a nonempty connected subset of $\mathbb{R}$, let $f_1, \dots, f_m$, and $g_1, \dots, g_n\colon I \to \mathbb{R}$ be continuous functions. Suppose that

(I) $\bigcup_{i=1}^{m} \Gamma(f_i) = \bigcup_{i=1}^{n} \Gamma(g_i)$,

(II) $\Gamma(f_1), \dots, \Gamma(f_m)$ are pairwise disjoint,

(III) $\Gamma(g_1), \dots, \Gamma(g_n)$ are pairwise disjoint.

Then, we have

(i) $m = n$,

(ii) There exists $\sigma \in S_m$ such that

$$(f_1, \dots, f_m) = (g_{\sigma(1)}, \dots, g_{\sigma(m)}). \tag{3.10}$$

Proof. From hypothesis (I), we obtain

$$\Gamma(f_1), \dots, \Gamma(f_m) \subset \bigcup_{i=1}^{n} \Gamma(g_i), \tag{3.11}$$

thus, by hypothesis (III) and proposition 3.2, we see that there exists a mapping $\sigma\colon \{1, \dots, m\} \to \{1, \dots, n\}$ such that

$$\Gamma(f_i) = \Gamma(g_{\sigma(i)}) \tag{3.12}$$

for all $i \in \{1, \ldots, m\}$. By hypothesis (II), on the other hand, $\Gamma(g_{\sigma(1)}), \ldots, \Gamma(g_{\sigma(m)})$ are pairwise disjoint. This implies that $\sigma(1), \ldots, \sigma(m)$ are all distinct from each other and hence that σ is an injection. Thus, $m \leqslant n$.

Similarly, starting from

$$\Gamma(g_1), \ldots, \Gamma(g_n) \subset \bigcup_{i=1}^{m} \Gamma(f_i), \tag{3.13}$$

we get $n \leqslant m$.

Therefore, $m = n$, and σ is a bijection. □

Proposition 3.4 (Glueing Tool 4). Let I be a nonempty connected subset of $\mathbb{R}$, let $f_1, \ldots, f_m$, and $g_1, \ldots, g_n$: $I \to \mathbb{R}$ be continuous functions. Suppose that

(I) $\bigcup_{i=1}^{m} \Gamma(f_i) = \bigcup_{i=1}^{n} \Gamma(g_i)$,

(II) $\Gamma(f_1), \ldots, \Gamma(f_m)$ are pairwise non-identical,

(III) $\Gamma(g_1), \ldots, \Gamma(g_n)$ are pairwise disjoint.

Then, $\Gamma(f_1), \ldots, \Gamma(f_m)$ are pairwise disjoint.

Proof. Let $i, j \in \{1, \ldots, m\}$ be such that $i \neq j$. Then, (II) implies that

$$\Gamma(f_i) \neq \Gamma(f_j). \tag{3.14}$$

We have to show that

$$\Gamma(f_i) \cap \Gamma(f_j) = \emptyset. \tag{3.15}$$

By the obvious relation

$$\Gamma(f_i) \subset \bigcup_{i=1}^{m} \Gamma(f_i), \tag{3.16}$$

and by (I), we have

$$\Gamma(f_i) \subset \bigcup_{i=1}^{n} \Gamma(g_i). \tag{3.17}$$

So, by applying proposition 3.2, we see that there exists a unique $k \in \{1, \ldots, n\}$ such that

$$\Gamma(f_i) = \Gamma(g_k). \tag{3.18}$$

Similarly, we see that there exists a unique $l \in \{1, \ldots, n\}$ such that

$$\Gamma(f_j) = \Gamma(g_l). \tag{3.19}$$

Now the following two cases are possible:

(i) $k = l$ and $\Gamma(g_k) = \Gamma(g_l)$,

(ii) $k \neq l$ and $\Gamma(g_k) \cap \Gamma(g_l) = \emptyset$.

But, (i) together with (3.18) and (3.19) would imply $\Gamma(f_i) = \Gamma(f_j)$, which contradicts with (3.14). Thus, (ii) must be true, which together with (3.18) and (3.19) imply (3.15). □

Proposition 3.5 (Glueing Tool 5). Let $A, B, a, b, s \in \mathbb{R}$ be such that $A < a < s < b < B$. Let $\lambda_1, \ldots, \lambda_m$ be real-valued continuous functions defined on $]A, B[$. Let $h_1, \ldots, h_n$ be real analytic functions defined on $]a, b[$ such that $\Gamma(h_1), \ldots, \Gamma(h_n)$ are pairwise non-identical. For each $\sigma \in S_m$, define functions $\lambda_1^\sigma, \ldots, \lambda_m^\sigma$: $]A, B[\to \mathbb{R}$ by

$$\lambda_i^\sigma(x) = \begin{cases} \lambda_i(x) & \text{if } x \in]A, s] \\ \lambda_{\sigma(i)}(x) & \text{if } x \in]s, B[. \end{cases} \tag{3.20}$$

Suppose that

(I) $\Gamma(\lambda_1 |]a, s[), \ldots, \Gamma(\lambda_m |]a, s[)$ are pairwise disjoint,

(II) $\Gamma(\lambda_1 |]s, b[), \ldots, \Gamma(\lambda_m |]s, b[)$ are pairwise disjoint,

(III) $\lambda_1, \ldots, \lambda_m$ are all real analytic on the interval $]A, s[$,

(IV) $\lambda_1, \ldots, \lambda_m$ are all real analytic on the interval $]s, B[$,

(V) $\bigcup_{i=1}^{m} \Gamma(\lambda_i |]a, b[) = \bigcup_{i=1}^{n} \Gamma(h_i)$.

Then, there exists a unique $\sigma \in S_m$ such that $\lambda_1^\sigma, \ldots, \lambda_m^\sigma$ are all real analytic on $]A, B[$.

Proof. By considering the intersection of $]a, s[\times \mathbb{R}$ and the set given by (V), we get

$$\bigcup_{i=1}^{m} \Gamma(\lambda_i |]a, s[) = \bigcup_{i=1}^{n} \Gamma(h_i |]a, s[). \tag{3.21}$$

Note that $\Gamma(h_1 |]a, s[), \ldots, \Gamma(h_n |]a, s[)$ are pairwise non-identical and that $\lambda_i |]a, s[$ and $h_i |]a, s[$ are all continuous. Thus, using propositions 3.3 and 3.4, we see that

$$m = n \tag{3.22}$$

and that there exists a $\mu \in S_m$ such that

$$\lambda_i |]a, s[= h_{\mu(i)} |]a, s[\tag{3.23}$$

holds for all $i \in \{1, \dots, m\}$.

Similarly, by considering the intersection of $]s, b[\times \mathbb{R}$ and the set given by (V), we see that there exists a $\nu \in S_m$ such that

$$\lambda_i |]s, b[= h_{\nu(i)} |]s, b[\tag{3.24}$$

holds for all $i \in \{1, \dots, m\}$. Set

$$\sigma = \mu \circ \nu^{-1}. \tag{3.25}$$

Then, by the group property of S_m, we have

$$\lambda_{\sigma(i)} |]s, b[= h_{\mu(i)} |]s, b[\tag{3.26}$$

for all $i \in \{1, \dots, m\}$.

Now recall the definition of λ_i^σ, and notice that equalities (3.23) and (3.26) and the easily verifiable fact that $\lambda_i(s) = h_{\mu(i)}(s)$ for all $i \in \{1, \dots, m\}$ imply that

$$\lambda_i^\sigma |]a, b[= h_{\mu(i)} \tag{3.27}$$

for all $i \in \{1, \dots, m\}$. Hence in view of the assumption that $h_1, \dots, h_m$ are all real analytic on $]a, b[$, $\lambda_1^\sigma, \dots, \lambda_m^\sigma$ are all real analytic on $]a, b[$. On the other hand, by the definition of λ_i^σ, (III), and (IV), it is easily seen that $\lambda_1^\sigma, \dots, \lambda_m^\sigma$ are all real analytic on $]A, s[$ and $]s, B[$, and hence that $\lambda_1^\sigma, \dots, \lambda_m^\sigma$ are all real analytic on $]A, B[$. Thus, we have proved that there exists a $\sigma \in S_m$ such that $\lambda_1^\sigma, \dots, \lambda_m^\sigma$ are all real analytic on $]A, B[$. The uniqueness of such a σ directly follows from (II), (III), and (IV). □

4. A solution of the problem of reduction via the basic piecewise monotone theorem and the glueing tools

First, we introduce the key theorem for the solution of our problem, whose proof will be given at the end of this section.

Theorem 4.1 (Basic Piecewise Monotone Theorem, BPMT). Let $a, b \in \mathbb{R}$ with $a < b$ and let $I = [a, b]$. Let $p \in C^\omega(I)[\lambda]$ be a monic polynomial of degree $q \in \mathbb{Z}^+$ given by

$$p = \lambda^q + c_1\lambda^{q-1} + \cdots + c_q. \tag{4.1}$$

Suppose that for any $\theta \in I$, the polynomial

$$\mathrm{Ev}_\theta(p) = \lambda^q + c_1(\theta)\lambda^{q-1} + \cdots + c_q(\theta) \tag{4.2}$$

over the field $\mathbb{R}$ has q real roots. Let $f := \mathrm{Fn}(p)$, i.e., define $f\colon I \times \mathbb{R} \to \mathbb{R}$ by

$$f(\theta, \lambda) = \lambda^q + c_1(\theta)\lambda^{q-1} + \cdots + c_q(\theta). \tag{4.3}$$

Let $u, v \in \mathbb{R}$ with $a \leqslant u < v \leqslant b$ and let $J = [u, v]$. Let $l \in C^{\omega *}(J)$. Suppose that

$$f(\theta, l(\theta)) = 0 \tag{4.4}$$

for all $\theta \in J$. Then, $l \in C^{PM}(J)$.

Second, we introduce the following proposition 4.1 (LAP2), which is seemingly a stronger version of LAP1, but is equivalent to LAP1 as proposition 4.2 indicates.

Proposition 4.1 (Local Analyticity Proposition, Version 2, LAP2). Let $a, b \in \mathbb{R}$ with $a < b$ and let $I = [a, b]$. Let $p \in C^{\omega}(I)[\lambda]$ be a monic polynomial of degree $q \in \mathbb{Z}^+$ given by

$$p = \lambda^q + c_1\lambda^{q-1} + \cdots + c_q. \tag{4.5}$$

Suppose that for any $\theta \in I$, the polynomial

$$\mathrm{Ev}_\theta(p) = \lambda^q + c_1(\theta)\lambda^{q-1} + \cdots + c_q(\theta) \tag{4.6}$$

over the field $\mathbb{R}$ has q real roots.

Define $f\colon I \times \mathbb{R} \to \mathbb{R}$ by

$$f(\theta, \lambda) = \lambda^q + c_1(\theta)\lambda^{q-1} + \cdots + c_q(\theta). \tag{4.7}$$

Then, for any $\theta \in]a, b[$ there exist $\varepsilon > 0, n \in \mathbb{Z}^+$, and $h_1, \ldots, h_n \in H_r(\theta-\varepsilon, \theta+\varepsilon)$ such that

$$f^{-1}(0) \cap (]\theta - \varepsilon, \theta + \varepsilon[\times \mathbb{R}) = \bigcup_{i=1}^{n} \Gamma(h_i). \tag{4.8}$$

Proposition 4.2 The LAP1 implies the LAP2.

Proof. Assume LAP1. Fix an arbitrarily given $\Theta \in]a, b[$. Let $\Lambda_1 < \cdots < \Lambda_m$ be all the roots of the equation $\mathrm{Ev}_\Theta(p) = 0$, in other words, let $\Lambda_1 < \cdots < \Lambda_m$ be such that

$$\{\Lambda_1, \ldots, \Lambda_m\} = \{\lambda \in \mathbb{R}\colon f(\Theta, \lambda) = 0\}. \tag{4.9}$$

By LAP1 and the continuity of analytic functions, one easily verifies that there exist $\varepsilon, \delta_1, \ldots, \delta_m \in \mathbb{R}^+$, $N_1, \ldots, N_m \in \mathbb{Z}^+$, $h_1^1, \ldots, h_{N_1}^1, \ldots, h_1^m, \ldots, h_{N_m}^m \in H_r(\Theta - \varepsilon, \Theta + \varepsilon)$ such that

$$h_1^j(\Theta) = \cdots = h_{N_j}^j(\Theta) = \Lambda_j \tag{4.10}$$

for all $j \in \{1, \ldots, m\}$, and such that

$$E_j :=]\Theta - \varepsilon, \Theta + \varepsilon[\times]\Lambda_j - \delta_j, \Lambda_j + \delta_j[\tag{4.11}$$

$$D_j :=]\Theta - \varepsilon, \Theta + \varepsilon[\times]\Lambda_j - \delta_j/2, \Lambda_j + \delta_j/2[\tag{4.12}$$

have the following properties:

(i) $f^{-1}(0) \cap E_j = \bigcup_{i=1}^{N_j} \Gamma(h_i^j)$ for all $j \in \{1, \ldots, m\}$,

(ii) $f^{-1}(0) \cap (E_j - D_j) = \varnothing$ for all $j \in \{1, \ldots, m\}$.

Hence, by (i), for the proof of the proposition, it suffices to show that

$$f^{-1}(0) \cap \big(]\Theta - \varepsilon, \Theta + \varepsilon[\times \mathbb{R}\big) \subset f^{-1}(0) \cap \left(\bigcup_{j=1}^{m} E_j\right), \tag{4.13}$$

$$f^{-1}(0) \cap \big(]\Theta - \varepsilon, \Theta + \varepsilon[\times \mathbb{R}\big) \supset f^{-1}(0) \cap \left(\bigcup_{j=1}^{m} E_j\right). \tag{4.14}$$

Since (4.14) is evident by the definition of E_j, we shall subsequently show that (4.13) is true. For each $\theta \in I$, let $\lambda_s(\theta)$ denote the sth root of the polynomial $\mathrm{Ev}_\theta(p)$. Recall proposition 3.1 (Glueing Tool 1), which asserts that real-valued function $\lambda_s : \theta \mapsto \lambda_s(\theta)$ defined on I is continuous for all $s \in \{1, \ldots, q\}$. For each $s \in \{1, \ldots, q\}$, let γ_s denote the graph of the restriction of λ_s to the interval $]\Theta - \varepsilon, \Theta + \varepsilon[$:

$$\gamma_s := \Gamma\big(\lambda_s |]\Theta - \varepsilon, \Theta + \varepsilon[\big). \tag{4.15}$$

Then, by the definitions of λ_s and γ_s,

$$f^{-1}(0) \cap (]\Theta - \varepsilon, \Theta + \varepsilon[\times \mathbb{R}) = \bigcup_{s=1}^{q} \gamma_s, \tag{4.16}$$

and (4.13) is equivalent to the relation

$$\bigcup_{s=1}^{q} \gamma_s \subset f^{-1}(0) \cap \left(\bigcup_{j=1}^{m} E_j\right). \tag{4.17}$$

Note that the left-hand side of (4.17) is obviously included in $f^{-1}(0)$, hence it remains to show that

$$\bigcup_{s=1}^{q} \gamma_s \subset \bigcup_{j=1}^{m} E_j. \tag{4.18}$$

Given an $s \in \{1, \dots, q\}$, let $t \in \{1, \dots, m\}$ be such that

$$(\Theta, \lambda_s(\Theta)) \in E_t, \tag{4.19}$$

which is obviously true for some $t \in \{1, \dots, m\}$. We claim that

$$\gamma_s \subset E_t. \tag{4.20}$$

Suppose that (4.20) were not true, then there would exist a $\xi \in]\Theta-\varepsilon, \Theta+\varepsilon[$ such that $(\xi, \lambda_s(\xi)) \notin E_t$, i.e., such that

$$\lambda_s(\xi) \notin]\Lambda_t - \delta_t, \Lambda_t + \delta_t[. \tag{4.21}$$

By the continuity of λ_s and by the Intermediate Value Theorem, (4.21) implies that

$$\gamma_s \cap (E_t - D_t) \neq \emptyset. \tag{4.22}$$

Now from (ii) and the obvious relation $\gamma_s \subset f^{-1}(0)$, we get

$$\gamma_s \cap (E_t - D_t) = \emptyset, \tag{4.23}$$

which contradicts with (4.22). Therefore, our claim (4.20) is true, and this completes the proof. □

Next, we introduce a weaker version of PML2, which is henceforth signified as PML2ω for short.

Lemma PML2ω. Let $a, b \in \mathbb{R}$ with $a < b$ and let $I = [a, b]$. Let $p \in C^{\omega}(I)[\lambda]$ be a monic polynomial of degree $q \in \mathbb{Z}^+$ given by

$$p = \lambda^q + c_1\lambda^{q-1} + \cdots + c_q. \tag{4.24}$$

Suppose that for any $\theta \in I$, the polynomial

$$\mathrm{Ev}_\theta(p) = \lambda^q + c_1(\theta)\lambda^{q-1} + \cdots + c_q(\theta) \tag{4.25}$$

over the field $\mathbb{R}$ has q real roots. Consider p as an element of $C^{\omega*}(I)[\lambda]$, then p can be factored into first degree monic polynomials:

$$p = (\lambda - d_1)(\lambda - d_2) \cdots (\lambda - d_q), \tag{4.26}$$

where

$$d_1, \dots, d_q \in C^{\omega*}(I). \tag{4.27}$$

Proposition 4.3 The LAP2 implies the PML2ω.

Proof. Assume the validity of the LAP2. Under the assumption of the PML2ω, let $\mathscr{D}(\theta)$ denote the discriminant of $\mathrm{Ev}_\theta(p)$. If $\mathscr{D}(\theta) \equiv 0$ on I, a factorization of p into irreducible factors has multiple non-constant factors. Using this fact, one easily sees that the proof of the proposition is reduced to the case when $\mathscr{D}(\theta) \not\equiv 0$ on I, which we henceforth assume. (*Note*: Suppose that the monic polynomial p has a factorization: $p = p_1 \dots p_k$, where $p_1, \dots, p_k \in C^\omega(I)[\lambda]$ and that c_{0i} is the leading coefficient of polynomial p_i for $i \in \{1, \dots, m\}$. Then, we have $c_{01} \dots c_{0k} = (c_{01} \dots c_{0k})^{-1} = c_{01}^{-1} \dots c_{0k}^{-1} = 1$. Hence, given a factorization of the monic polynomial p: $p = p_1 \dots p_k$, the factorization can be put into the form: $p = (c_{01}^{-1} p_1) \dots (c_{0k}^{-1} p_k)$, where each $c_{0i}^{-1} p_i$ is a monic polynomial. Also notice that for any $\theta \in I$, $\mathrm{Ev}_\theta(p) = \mathrm{Ev}_\theta(c_{01}^{-1} p_1) \dots \mathrm{Ev}_\theta(c_{0k}^{-1} p_k)$ and that the polynomial $\mathrm{Ev}_\theta(c_{0i}^{-1} p_i)$ over the field $\mathbb{R}$ has q_i real roots for any $\theta \in I$, where $q_i := \deg(\mathrm{Ev}_\theta(c_{0i}^{-1} p_i)), i \in \{1, \dots, k\}$.)

Let K denote the set of zeros of the real-valued function $\mathscr{D}: \theta \mapsto \mathscr{D}(\theta)$ defined on I. Since $\mathscr{D}$ is real analytic on the compact interval I, we see that K is a finite set.

For each $\theta \in I$, denote the ith root of the polynomial $\mathrm{Ev}_\theta(p)$ by $\lambda_i(\theta)$. First, recall proposition 3.1 (Glueing Tool 1), which asserts that real-valued function λ_i: $\theta \mapsto \lambda_i(\theta)$ defined on I is continuous for all $i \in \{1, \dots, q\}$. Next note that $i \neq j$ implies that

$$\lambda_i(\theta) \neq \lambda_j(\theta) \tag{4.28}$$

for all $\theta \in I - K$ because $\mathscr{D}(\theta) \neq 0$ for all $\theta \in I - K$.

By the Implicit Function Theorem, we easily demonstrate that all the λ_i are real analytic on $]a, b[- K$. (*Remark*: This fact also follows directly from some theoretical tools constructed in part III of this series of articles.)

By the fact that all the λ_i are real analytic on $]a, b[- K$ and continuous on $[a, b]$, and by LAP2 and proposition 3.5 (Glueing Tool 5), one easily verifies the conclusion of the PML2ω is true if LAP2 is true. □

Now we can state:

Solution of our problem of reduction. We know that

$$\text{LAP1} \Rightarrow \text{LAP2} \Rightarrow \text{PML2}\omega, \tag{4.29}$$

where the implications have been verified using the glueing tools developed in section 3.

On the other hand, we easily see that

$$\text{PML2}\omega \ \& \ \text{BPMT} \Rightarrow \text{PML2} \Rightarrow \mathscr{G}\text{BT}. \tag{4.30}$$

Hence, for the affirmative solution of our problem of reduction, it remains to prove the BPMT. After preparing the following propositions 4.4 and 4.5, we demonstrate the BPMT.

Proposition 4.4 Let D be a unique factorization domain. Let p_1, p_2 be two polynomials given by

$$p_1 = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \in D[x] \quad (a_0 \neq 0), \tag{4.31}$$

$$p_2 = b_0 x^m + b_1 x^{m-1} + \cdots + b_m \in D[x]. \tag{4.32}$$

Let $R(p_1, p_2)$ stand for the resultant of the polynomials p_1, p_2. Then p_1 and p_2 have a common non-constant factor if and only if

$$R(p_1, p_2) = 0. \tag{4.33}$$

(For the proof of proposition 4.4, see, e.g. Ref. [10].)

Proposition 4.5 Let D be a unique factorization domain of characteristic zero. Let p be a polynomial given by

$$p = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \in D[x] \quad (a_0 \neq 0). \tag{4.34}$$

Then a factorization of the polynomial p into irreducible factors has multiple non-constant factors if and only if the discriminant of p vanishes

$$\mathscr{D}(p) = R(p, p') = 0. \tag{4.35}$$

(For the proof of proposition 4.5, see, e.g. Ref. [11].)

We are now ready to provide

Proof of theorem 4.1 (BPMT). For the proof of the theorem, we may and do assume that $I = J$ without loss of generality.

Define $w \in C^{\omega}(I)$ by

$$w = R\left(p, \frac{\partial p}{\partial \theta}\right), \tag{4.36}$$

where

$$\frac{\partial p}{\partial \theta} := \frac{\partial c_1}{\partial \theta}\lambda^{q-1} + \frac{\partial c_2}{\partial \theta}\lambda^{q-2} + \cdots + \frac{\partial c_q}{\partial \theta} \in C^{\omega}(I)[\lambda]. \tag{4.37}$$

Let $W \in C^{\omega}(I)[\lambda]$ be the 0th degree polynomial defined by

$$W = w\lambda^0. \tag{4.38}$$

That is, we consider w as an element of $C^{\omega}(I)[\lambda]$, denoted W. By the fundamental property of the resultant, W can be written as

$$W = Ap + B\frac{\partial p}{\partial \theta}, \tag{4.39}$$

where A and B are some elements in $C^{\omega}(I)[\lambda]$. Now apply the ring homomorphism Fn to the both sides:

$$\mathrm{Fn}(W) = \mathrm{Fn}(A)\mathrm{Fn}(p) + \mathrm{Fn}(B)\mathrm{Fn}\left(\frac{\partial p}{\partial \theta}\right). \tag{4.40}$$

Set $u := \mathrm{Fn}(A)$, $v := \mathrm{Fn}(B)$, and note that $f = \mathrm{Fn}(p)$, $\frac{\partial f}{\partial \theta} = \mathrm{Fn}\left(\frac{\partial p}{\partial \theta}\right)$, and that $w(\theta) = \mathrm{Fn}(W)(\theta, \lambda)$ for all $(\theta, \lambda) \in I \times \mathbb{R}$. Thus, the equality

$$w(\theta) = u(\theta, \lambda) f(\theta, \lambda) + v(\theta, \lambda)\frac{\partial f}{\partial \theta}(\theta, \lambda) \tag{4.41}$$

holds for all $(\theta, \lambda) \in]a, b[\times \mathbb{R}$. Substituting $l(\theta)$ for λ in the right-hand side of this equality, we then have

$$w(\theta) = u(\theta, l(\theta)) f(\theta, l(\theta)) + v(\theta, l(\theta))\frac{\partial f}{\partial \theta}(\theta, l(\theta)) \tag{4.42}$$

for all $\theta \in]a, b[$. Differentiating the real-analytic function $\theta \mapsto f(\theta, l(\theta))$ defined on $]a, b[$, and recalling (4.4), we have

$$\frac{\partial f}{\partial \theta}(\theta, l(\theta)) + \frac{\partial f}{\partial \lambda}(\theta, l(\theta)) l'(\theta) = 0 \tag{4.43}$$

for all $\theta \in]a, b[$.

Now notice that (4.4), (4.42), and (4.43) imply

$$\{\theta \in]a, b[: l'(\theta) = 0\} \subset \{\theta \in]a, b[: w(\theta) = 0\}. \tag{4.44}$$

Since $w \in C^{\omega}(I)$, the set of zeroes of w on $[a, b]$ is either a finite set or $[a, b]$ itself. We consider these two cases separately below:

Case 1: If $\{\theta \in [a, b] : w(\theta) = 0\}$ is a finite set, then $\{\theta \in]a, b[: l'(\theta) = 0\}$ is evidently a finite set and $l \in C^{\mathrm{PM}}(I)$.

Case 2: If $\{\theta \in [a, b] : w(\theta) = 0\} = [a, b]$, then $w = 0 \in C^{\omega}(I)$ and the proposition 4.4 implies that p and $\frac{\partial p}{\partial \theta}$ have a common non-constant factor.

Now recalling the fact that p is a monic polynomial, clearly we have

$$\deg\left(\frac{\partial p}{\partial \theta}\right) < \deg(p). \tag{4.45}$$

Given a factorization of

$$p = \prod_{j=1}^{m} p_j, \tag{4.46}$$

where all the p_j are irreducible monic polynomials, the equality

$$\mathrm{Ev}_\theta(p) = \prod_{j=1}^{m} \mathrm{Ev}_\theta(p_j) \tag{4.47}$$

holds for each $\theta \in I$. Let

$$q_j := \deg(p_j). \tag{4.48}$$

Since $\mathbb{R}[\lambda]$ is a UFD and since $\mathrm{Ev}_\theta(p)$ can be factored into $q(= \deg(p))$ monic linear factors in $\mathbb{R}[\lambda]$ by the hypothesis of the proposition, $\mathrm{Ev}_\theta(p_j)$ can be factored into q_j monic linear factors in $\mathbb{R}[\lambda]$.

We claim that there exists a $j \in \{1, \dots, m\}$ such that

$$\mathrm{Fn}(p_j)(\theta, l(\theta)) = 0 \tag{4.49}$$

for all $\theta \in I$. In fact by applying Fn to (4.46), and evaluating at $(\theta, l(\theta))$, we have

$$\mathrm{Fn}(p)(\theta, l(\theta)) = \prod_{j=1}^{m} \mathrm{Fn}(p_j)(\theta, l(\theta)) \tag{4.50}$$

for all $\theta \in I$. Let $p_0 := p$, and for each $j \in \{0, \dots, q\}$ define $g_j \in C^{\omega*}(I)$ by

$$g_j(\theta) = \mathrm{Fn}(p_j)(\theta, l(\theta)). \tag{4.51}$$

Recalling the hypothesis of the proposition, note that

$$g_0(\theta) = \prod_{j=1}^{m} g_j(\theta) = 0 \tag{4.52}$$

for all $\theta \in I$, so that we have

$$I = \bigcup_{j=1}^{m} g_j^{-1}(0). \tag{4.53}$$

Pick any compact subinterval I_0 of $]a, b[$, say, $I_0 := [a + (b-a)/3, b - (b-a)/3]$. Then, equality (4.53) implies that for at least one $j \in \{1, \dots, m\}$, the set of zeros of $g_j|I_0$ is infinite, hence g_j vanishes on the compact interval I_0. So, for some $j \in \{1, \dots, m\}$, g_j, which is real-analytic on $]a, b[$, vanishes on $]a, b[$ and hence, by continuity, on the whole domain $I = [a, b]$.

Thus, we may and do henceforth assume that p is irreducible.

But, under this assumption, by inequality (4.45), p and $\frac{\partial p}{\partial \theta}$ have a common non-constant factor, only if

$$\frac{\partial p}{\partial \theta} = \frac{\partial c_1}{\partial \theta}\lambda^{q-1} + \frac{\partial c_2}{\partial \theta}\lambda^{q-2} + \cdots + \frac{\partial c_q}{\partial \theta} = 0 \in C^{\omega}(I)[\lambda], \tag{4.54}$$

i.e., only if $c_1, \ldots, c_q$ are constant functions. If so, we easily see that there exist constant functions $d_1, \ldots, d_q \in C^{\omega}(I)$ such that

$$p = (\lambda - d_1)(\lambda - d_2) \cdots (\lambda - d_q). \tag{4.55}$$

Note that q must be 1, since p is irreducible by our assumption. We now see that l equals a constant function d_1 and hence $l \in C^{PM}(I)$. □

We have established the BPMT, and hence the following implications are true:
LAP1 $\Rightarrow$ PML2 $\Rightarrow$ $\mathscr{G}$ Boundedness Theorem $\Rightarrow$ Special Functional ALT $\Rightarrow$ Functional ALT $\Rightarrow$ the Fukui conjecture.

By applying the theory of algebraic curves and by recalling techniques used in perturbation theory [9], it is seen that the LAP1 admits a proof using resolution of singularities and related methods. In part III of this series, we shall provide a detailed proof of the LAP1 in conjunction with the repeat space theory [1–8].

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References

[1] S. Arimoto, M. Spivakovsky, K.F. Taylor and P.G. Mezey, J. Math. Chem. 37 (2005) 75.
[2] S. Arimoto, J. Math. Chem. 34 (2003) 253.
[3] S. Arimoto, J. Math. Chem. 34 (2003) 259.
[4] S. Arimoto, J. Math. Chem. 34 (2003) 287.
[5] S. Arimoto and K. Fukui, IFC Bull. (1998) 7. (available at http://duke.usask.ca/~arimoto/)
[6] S. Arimoto, K. Fukui, P. Zizler, K.F. Taylor and P.G. Mezey, Int. J. Quant. Chem. 74 (1999) 633.
[7] S. Arimoto, M. Spivakovsky, H. Ohno, P. Zizler, K.F. Taylor, T. Yamabe and P.G. Mezey, Int. J. Quant. Chem. 84 (2001) 389.
[8] S. Arimoto, M. Spivakovsky, H. Ohno, P. Zizler, R.A. Zuidwijk, K.F. Taylor, T. Yamabe and P.G. Mezey, Int. J. Quant. Chem. 97 (2004) 765.
[9] F. Rellich, *Perturbation Theory of Eigenvalue Problems* (Gordon and Breach, New York, 1969).
[10] P.A. Griffiths, *Introduction to Algebraic Curves* (American Mathematical Society, Providence, 1989).
[11] R.J. Walker, *Algebraic Curves* (Princeton University Press, Princeton, 1950).